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Classical and Quantum Action-Phase Variables for Time-Dependent Oscillators

Sang Pyo Kim* and Don N. Page†‡

Theoretical Physics Institute

Department of Physics

University of Alberta

Edmonton, Alberta, Canada T6G 2J1

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Abstract

For a time-dependent classical quadratic oscillator we introduce pairs of real and complex invariants that are linear in position and momentum. Each pair of invariants realize explicitly a canonical transformation from the phase space to the invariant space, in which the action-phase variables are defined. We find the action operator for the time-dependent quantum oscillator via the classical-quantum correspondence. Candidate phase operators conjugate to the action operator are discussed, but no satisfactory ones are found.

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*E-mail: spkim@phys.ualberta.ca

†E-mail: don@phys.ualberta.ca

‡Fellow, Cosmology and Gravity Program of the Canadian Institute for Advanced Research

I. INTRODUCTION

A time-independent classical oscillator is exactly solved when the evolution of the position and momentum in the phase space is completely known in terms of the initial data or constants of motion. A constant of motion is a function of the initial data in phase space, and the evolution of an oscillator is completely determined by any two independent constants. For any linear second-order system, such as a time-dependent classical oscillator, two invariants (constants of motion) are readily constructed from two independent classical solutions. The time-dependent quantum oscillator can be exactly solved in terms of the classical solutions, as the invariant operator, which was first introduced by Lewis and Riesenfeld and is quadratic in position and momentum, is known explicitly in terms of the solutions of the corresponding classical oscillator [1]. The invariant operator generates the Fock space of exact quantum states of the Schrödinger equation. Recently the time-dependent quantum oscillator has attracted much attention with the discovery of the geometric phase by Berry to study the nonadiabatic geometric phase for various quantum states, such as Gaussian, number, squeezed, or coherent states, which can be found exactly [2,3].

In spite of intensive study of time-dependent oscillators, the issue of action-phase (angle) variables remains not fully exploited classically and quantum mechanically. In particular, the quantum action-phase operators are still elusive and not completely resolved since it was raised by Dirac [4]. For a time-independent classical oscillator the Hamiltonian itself, being a constant of motion, turns out to be an action variable, and its conjugate is the phase (angle) variable. The transformation from the phase space to the action-phase space is a canonical transformation. A generating function that yields the action-phase variables can be found even for the time-dependent classical oscillator [5]. On the other hand, for a time-independent quantum oscillator many different methods and schemes have been introduced to define the action-phase operators [6]. Recently a candidate for a phase operator conjugate to the Hamiltonian (action) operator has been proposed for the time-independent quantum oscillator [7].

The aim of this paper is to find the canonical transformations from the phase space to the invariant spaces and to define the action-phase variables for a time-dependent quadratic oscillator. For that purpose we introduce pairs of classical invariants that are linear in position and momentum and are expressed in terms of the classical solutions, which realize explicitly the canonical transformation from the phase space to the invariant spaces. We find the most general three-parameter quadratic invariants formed from the linear ones, which exhaust all quadratic invariants and include the one by Lewis and Riesenfeld. In each invariant space we define the action variable, quadratic in position and momentum, and find its conjugate phase variable. Analogously by using pairs of linear invariant operators that constitute the annihilation and creation operators for the Fock space of exact quantum states [8], we find the action operator for the time-dependent quantum oscillator. The three-parameter invariant operator is transformed into the canonical form of a number operator through a unitary (Bogoliubov) transformation. Similarly the invariant operator obtained from one classical solution is transformed into the one obtained from another classical solution through a unitary transformation. A number eigenstate of one pair is a squeezed state of the corresponding one of another pair. We extend the current scheme for defining the phase operator to the time-dependent quantum oscillator.

The organization of the paper is as follows. In Sec. II we find pairs of linear invariants for the time-dependent classical oscillator and show the canonical transformation from the phase space to the invariant space. In each invariant space we define the action-phase variables. In Sec. III we find the action operator and define the phase operator conjugate to the action operator.

II. CLASSICAL ACTION-PHASE VARIABLES

We consider a time-dependent classical oscillator which is described by the Hamiltonian

$$H(t) = \frac{X(t)}{2}p^2 + \frac{Y(t)}{2}(pq + qp) + \frac{Z(t)}{2}q^2, \tag{1}$$

where $X(t), Y(t)$ and $Z(t)$ depend explicitly on time. An invariant for the Hamiltonian (1) is a constant of motion and obeys the equation

$$\frac{d}{dt}I(t) = \frac{\partial}{\partial t}I(t) + \{I(t), H(t)\}_{\rm PB} = 0, \tag{2}$$

where $\{\ ,\ \}_{\rm PB}$ denotes a Poisson bracket. To find any invariant it is necessary to find a pair of invariants, say, I_1 and I_2, which are linear in position and momentum and are independent each other, because the invariants satisfying Eq. (2) satisfy the multiplication property

$$\frac{\partial}{\partial t}(I_1 I_2) + \{I_1 I_2, H\}_{\rm PB} = \left[\frac{\partial}{\partial t}I_1 + \{I_1, H\}_{\rm PB}\right] I_2 + I_1\left[\frac{\partial}{\partial t}I_2 + \{I_2, H\}_{\rm PB}\right] = 0. \tag{3}$$

Hence any analytical function of the invariants I_1 and I_2 is also an invariant. For that purpose, we use two independent real solutions u_1 and u_2 to the classical equation of motion

$$\frac{d}{dt}\left(\frac{\dot{u}}{X}\right) + \left[XZ - Y^2 + \frac{\dot{X}Y - X\dot{Y}}{X}\right]\left(\frac{u}{X}\right) = 0, \tag{4}$$

where overdots denote d/dt. We require that these solutions are normalized to satisfy the Wronskian condition

$$\frac{1}{X}(u_1\dot{u}_2 - \dot{u}_1 u_2) = 1. \tag{5}$$

Then we associate two invariants to two independent solutions u_1 and u_2:

$$\begin{aligned} a_1(t) &= u_1(t)p - \frac{1}{X}[\dot{u}_1(t) - Yu_1(t)]q, \\ a_2(t) &= u_2(t)p - \frac{1}{X}[\dot{u}_2(t) - Yu_2(t)]q. \end{aligned} \tag{6}$$

Indeed a_1 and a_2 are canonical conjugate of each other, satisfying

$$\{a_1(t), a_2(t)\}_{\rm PB} = 1. \tag{7}$$

It should be remarked that there are as many pairs of independent invariants as pairs of independent solutions. These pairs of invariants constitute canonical conjugate pairs, satisfying Eq. (7), if Eq. (5) is satisfied. As any two independent solutions to Eq. (4), say v_1 and v_2, can be expressed in terms of the original pair, u_1 and u_2, these pairs are related to each other through a canonical transformation. In quantum theory the canonical transformation corresponds to a Bogoliubov transformation and has the meaning of squeezing the quantum states, which will be discussed Sec. III.

For a time-independent oscillator with constant X, Y and Z, the Hamiltonian (1) is an action variable. For the time-dependent oscillator of this paper, the Hamiltonian cannot become an action variable, since $\partial H(t)/\partial t \neq 0$. Instead we look for the most general quadratic invariant of the form

$$\mathcal{I} = \frac{1}{2}(B + A^* + A)a_1^2 + \frac{1}{2i}(A * -A)2a_1 a_2 + \frac{1}{2}(B - A^* - A)a_2^2, \tag{8}$$

where A is a complex constant and B is a real constant. In fact, Eq. (8) describes real three-parameter invariants that exhaust all real invariants that are quadratic in position and momentum. Through a canonical transformation, which is a rotation plus a scaling in the (a_1, a_2) space, Eq. (8) can be transformed into the canonical form

$$\mathcal{I}(t) = \frac{1}{2}[a_1^2(t) + a_2^2(t)]. \tag{9}$$

The canonical transformation has the physical meaning of squeezing in quantum theory, which will be discussed in Sec. III. Constant values of the action (9) depict circles in the (a_1, a_2) plane with polar coordinates defined through

$$a_1(t) = \sqrt{2\mathcal{I}(t)}\cos\theta(t), \quad a_2 = \sqrt{2\mathcal{I}(t)}\sin\theta(t). \tag{10}$$

The phase variable, simply given by

$$\theta(t) = \tan^{-1}\left[\frac{a_2(t)}{a_1(t)}\right] \tag{11}$$

up to arbitrary additive constant, is an invariant, and is the conjugate of the action (9):

$$\{\mathcal{I}(t), \theta(t)\}_{\rm PB} = 1. \tag{12}$$

It is then trivial to show that

$$\{\mathcal{I}(t), \tan\theta(t)\}_{\rm PB} = \sec^2\theta(t). \tag{13}$$

The transformations from (q, p) to (a_1, a_2) and from (a_1, a_2) to $(\mathcal{I}, \theta(t))$ are canonical, so their volume elements are preserved:

$$dq \wedge dp = da_1 \wedge da_2 = d\mathcal{I} \wedge d\theta. \tag{14}$$

The volume of the phase space with $\mathcal{I} \leq \mathcal{I}_0$ is then

$$\int_{\mathcal{I}\leq\mathcal{I}_0} dq \wedge dp = \int_{\mathcal{I}\leq\mathcal{I}_0} da_1 \wedge da_2 = \int_{\mathcal{I}\leq\mathcal{I}_0} d\mathcal{I} \wedge d\theta = 2\pi\mathcal{I}_0. \tag{15}$$

To make manifest the correspondence between the classical and quantum theory in Sec. III, we introduce another pair of invariants

$$a(t) = (i)\left\{u^*(t)p - \frac{1}{X(t)}[\dot{u}^*(t) - Y(t)u^*(t)]q\right\},$$
$$a^*(t) = (-i)\left\{u(t)p - \frac{1}{X(t)}[\dot{u}(t) - Y(t)u(t)]q\right\}, \tag{16}$$

where u is a complex solution to Eq. (4), normalized to satisfy the Wronskian condition

$$\frac{1}{X}(u\dot{u}^* - u^*\dot{u}) = i. \tag{17}$$

a and a^* are the complex conjugates of each other and satisfy the Poisson bracket

$$\{a(t), a^*(t)\}_{\mathrm{PB}} = -i. \tag{18}$$

The factor $-i$ is a consequence of complex nature of a and a^*. If $u(t) = [u_1(t) + iu_2(t)]/\sqrt{2}$, the complex invariants (16) are given by

$$a(t) = \frac{a_1(t) + ia_2(t)}{\sqrt{2}}, \quad a^*(t) = \frac{a_1(t) - ia_2(t)}{\sqrt{2}}. \tag{19}$$

Hence there is a canonical transformation from the phase space (q, p) to the invariant space (a, a^*). The action variable (9) is now given by

$$\begin{aligned}\mathcal{I}(t) &= a^*(t)a(t)\\ &= u^*u\left\{p + \left[\frac{Y}{X} - \frac{1}{X}\frac{d}{dt}\ln(u^*u)^{1/2}\right]q\right\}^2 + \frac{q^2}{4u^*u}.\end{aligned} \tag{20}$$

A general quadratic invariant of a and a^* can be transformed into the form (20) through a canonical transformation, which has the physical interpretation of squeezing in quantum theory, as will discussed in Sec. III.

As a^* is the complex conjugate of a, the phase of a can be defined by

$$a(t) = \sqrt{\mathcal{I}(t)}e^{i\theta_a(t)}, \quad a^*(t) = \sqrt{\mathcal{I}(t)}e^{-i\theta_a(t)}. \tag{21}$$

The polar form in Eq. (21) leads to the phase variable in the form

$$\theta_a(t) = \frac{i}{2}[\ln a^*(t) - \ln a(t)]. \tag{22}$$

The equivalence between the phases (11) and (22) can be shown using the identity

$$\theta(t) = \tan^{-1}\left[\frac{a_2(t)}{a_1(t)}\right] = \frac{i}{2}\ln\left[\frac{i + (a_2(t)/a_2(t))}{i - (a_2(t)/a_1(t))}\right] = \theta_a(t). \tag{23}$$

It follows that θ_a is the conjugate variable of $\mathcal{I}$:

$$\{\mathcal{I}(t), \theta_a(t)\}_{\mathrm{PB}} = 1. \tag{24}$$

After expressing q and p in (16) in terms of a and a^* by using (17), and writing the complex solution in a polar form

$$u(t) = \rho(t) e^{-i\theta_u(t)}, \tag{25}$$

we find

$$\begin{aligned} q &= ua + u^* a^* = 2\sqrt{\mathcal{I}}\rho\cos(\theta_a - \theta_u), \\ p &= \frac{1}{X}(\dot{u}a + \dot{u}^* a^*) - \frac{Y}{X}(ua + u^* a^*) = \frac{2\sqrt{\mathcal{I}}}{X}(\dot{\rho} - Y\rho)\cos(\theta_a - \theta_u) + \frac{\sqrt{\mathcal{I}}}{\rho}\sin(\theta_a - \theta_u). \end{aligned} \tag{26}$$

There is also another phase variable from the action (20). As Eq. (20) implies that constant $\mathcal{I}$ describes an ellipse in the (q, p) phase space, the phase variable can be defined by [3]

$$\begin{aligned} \cos\vartheta &= \sqrt{\frac{1}{4u^* u\mathcal{I}}}q, \\ \sin\vartheta &= \sqrt{\frac{u^* u}{\mathcal{I}}}\left\{p + \left[\frac{Y}{X} - \frac{1}{X}\frac{d}{dt}\ln(u^* u)^{1/2}\right]q\right\}. \end{aligned} \tag{27}$$

The phase (27) is the conjugate variable of the invariant through the generating function

$$\vartheta = \frac{\partial S}{\partial \mathcal{I}}, \tag{28}$$

where

$$S(\mathcal{I}, q; t) = \int_{q_0;\mathcal{I}=\text{fixed}}^{q} p(\mathcal{I}, q; t) dq. \tag{29}$$

Hence the two phase variables (27) and (21) are related by

$$\vartheta = \theta_a - \theta_u. \tag{30}$$

The canonical transformations from (q, p) to the invariant space of action-phase variables $(\mathcal{I}, \theta)$ or $(\mathcal{I}, \vartheta)$ preserve the phase volume

$$dq \wedge dp = d\mathcal{I} \wedge d\theta = d\mathcal{I} \wedge d\vartheta. \tag{31}$$

The area for $\mathcal{I}_0 \leq \mathcal{I}$ is

$$\int_{\mathcal{I}\leq\mathcal{I}_0} dq \wedge dp = \int_{\mathcal{I}\leq\mathcal{I}_0} d\mathcal{I} \wedge d\theta = \int_{\mathcal{I}\leq\mathcal{I}_0} d\mathcal{I} \wedge d\vartheta = 2\pi\mathcal{I}_0. \tag{32}$$

III. QUANTUM ACTION-PHASE OPERATORS

In quantum theory invariant operators obey the quantum Liouville-von Neumann equation

$$i\hbar \frac{\partial}{\partial t}\hat{I}(t) + [\hat{I}(t), \hat{H}(t)] = 0. \tag{33}$$

Corresponding to the pair of classical invariants (16), there are a pair of invariant operators [8]

$$\begin{aligned}\hat{a}(t) &= \frac{i}{\sqrt{\hbar}}\left[u^*(t)\hat{p} - \frac{1}{X(t)}[\dot{u}^*(t) - Y(t)u^*(t)]\hat{q}\right],\\ \hat{a}^\dagger(t) &= -\frac{i}{\sqrt{\hbar}}\left[u(t)\hat{p} - \frac{1}{X(t)}[\dot{u}(t) - Y(t)u(t)]\hat{q}\right],\end{aligned} \tag{34}$$

where u is a complex solution satisfying (5). Moreover, the invariant operators satisfy the standard commutation relation

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1. \tag{35}$$

There is the correspondence between the classical and quantum theory via $a \leftrightarrow \hat{a}$ and $a^* \leftrightarrow \hat{a}^\dagger$, except for some operator ordering problems. Just as the pair of classical invariants (16) are fundamental in the sense that any analytic function $F(a, a^*)$ is an invariant, the pair of invariant operators (34) are fundamental, since any analytic function $F(\hat{a}, \hat{a}^*)$ is also an invariant operator. This fact is guaranteed by the multiplication property of solutions of the Liouville-von Neumann equation,

$$i\hbar\frac{\partial}{\partial t}(\hat{I}_1\hat{I}_2) + [\hat{I}_1\hat{I}_2, \hat{H}] = \left(i\hbar\frac{\partial}{\partial t}\hat{I}_1(t) + [\hat{I}_1, \hat{H}]\right)\hat{I}_2 + \hat{I}_1\left(i\hbar\frac{\partial}{\partial t}\hat{I}_2 + [\hat{I}_2, \hat{H}]\right) = 0. \tag{36}$$

Hence the most general, quadratic, Hermitian, invariant operator is spanned by $\hat{a}^{\dagger 2}(t)$, $\hat{a}^\dagger(t)\hat{a}(t) + \hat{a}(t)\hat{a}^\dagger(t)$, and $\hat{a}^2(t)$ with one complex and one real parameter, which corresponds to the classical one (8).

The most general, quadratic, Hermitian invariant that can be constructed from the pair (34) has the form

$$\hat{I} = \frac{A}{2}\hat{a}^{\dagger 2}(t) + \frac{B}{2}[\hat{a}^\dagger(t)\hat{a}(t) + \hat{a}(t)\hat{a}^\dagger(t)] + \frac{A^*}{2}\hat{a}^2(t), \tag{37}$$

where A is a complex constant and B is a real constant. Another pair of invariant operators, say $\hat{b}(t)$ and $\hat{b}^\dagger(t)$, obtained by replacing $u(t)$ by an independent complex solution to Eq. (4), say $v(t)$, do not change the general form (37), since these are related to the original pair through the Bogoliuobov transformation

$$\begin{aligned}\hat{b}(t) &= \alpha\hat{a}(t) + \beta\hat{a}^\dagger(t),\\ \hat{b}^\dagger(t) &= \alpha^*\hat{a}^\dagger(t) + \beta^*\hat{a}(t),\end{aligned} \tag{38}$$

where α and β are determined by

$$
\begin{aligned}
v(t) &= \alpha^* u(t) - \beta^* u^*(t), \\
v^*(t) &= \alpha u^*(t) - \beta u(t).
\end{aligned} \tag{39}
$$

A unitary (Bogoliubov) transformation can be found,

$$
\begin{aligned}
\hat{a}(t) &= \cosh r \hat{\tilde{a}}(t) + \sinh r e^{i\delta} \hat{\tilde{a}}^\dagger(t) \\
&= \hat{S}(r,\delta) \hat{\tilde{a}}(t) \hat{S}^\dagger(r,\delta), \\
\hat{a}^\dagger(t) &= \cosh r \hat{\tilde{a}}^\dagger(t) + \sinh r e^{-i\delta} \hat{\tilde{a}}(t) \\
&= \hat{S}(r,\delta) \hat{\tilde{a}}^\dagger(t) \hat{S}^\dagger(r,\delta),
\end{aligned} \tag{40}
$$

where the unitary operator is a squeezing operator

$$
\hat{S}(z) = \exp\left[\frac{1}{2}\{z^* \hat{\tilde{a}}^2(t) - z \hat{\tilde{a}}^\dagger(t)\}\right], \quad z = r e^{\delta + \pi}, \tag{41}
$$

which transforms (37) to a canonical form

$$
\hat{\mathcal{I}}(t) = \frac{\tilde{B}}{2}[\hat{\tilde{a}}^\dagger(t)\hat{\tilde{a}}(t) + \hat{\tilde{a}}(t)\hat{\tilde{a}}^\dagger(t)]. \tag{42}
$$

Here the squeezing parameter r, phase δ, and the coefficient $\tilde{B}$ are determined by

$$
\begin{aligned}
e^{i\delta} \tanh r &= \frac{1}{A^*}[-B \pm \sqrt{B^2 - A^* A}], \\
\tilde{B} &= \frac{1}{2}[A \sinh 2r e^{-i\delta} + B \cosh 2r + A^* \sinh 2r e^{i\delta}].
\end{aligned} \tag{43}
$$

The physical meaning of the unitary transformation (38) is that the number state of $\hat{b}^\dagger(t)\hat{b}(t)$ is a squeezed state of the corresponding one of $\hat{a}^\dagger(t)\hat{a}(t)$. Similarly the number state of $\hat{a}^\dagger(t)\hat{a}(t)$ is a squeezed state of the corresponding one of $\hat{\tilde{a}}^\dagger(t)\hat{\tilde{a}}(t)$.

Without losing generality, we shall use the quantum action operator

$$
\hat{\mathcal{I}}(t) = \hbar \hat{a}^\dagger(t)\hat{a}(t) = \hbar \hat{n}(t), \tag{44}
$$

which corresponds to the action variable (9). The invariant operator (44) is nothing but a number operator. Each eigenstate of $\hat{\mathcal{N}}$ is a number state

$$
\hat{n}(t)|n,t\rangle = n|n,t\rangle. \tag{45}
$$

The wave function for the number state defined by Eq. (45), up to an arbitrary constant phase factor, is

$$
\Psi_n(q,t) = \frac{1}{\sqrt{(2\hbar)^n n!}}\left(\frac{1}{2\pi\hbar u^* u}\right)^{1/4}\left(\frac{u}{\sqrt{u^* u}}\right)^{(2n+1)/2} H_n\left(\frac{q}{\sqrt{2\hbar u^* u}}\right)\exp\left[\frac{i}{2\hbar X}\left(\frac{\dot{u}^*}{u^*} - Y\right)q^2\right], \tag{46}
$$

where H_n is the Hermite polynomial. It satisfies the time-dependent Schrödinger equation. In the sense that one finds the Fock space of exact quantum states of the Schrödinger equation, even the time-dependent quantum oscillator (1) is exactly solvable in terms of the complex classical solution $u(t)$.

Finally we turn to the phase operator conjugate to the action operator (44). A phase operator should satisfy the commutation relation

$$[\hat{\mathcal{I}}(t), \hat{\theta}(t)] = i\hbar, \tag{47}$$

where θ is restricted to $(0, 2\pi)$. The commutation relation leads to the Lerner criterion

$$[e^{i\hat{\theta}(t)}, \hat{\mathcal{I}}(t)] = \hbar e^{i\hat{\theta}(t)}. \tag{48}$$

There are many schemes to define the phase operator, and each method has its own advantage and disadvantage [6]. As there is a formal similarity between time-independent and time-dependent quantum oscillators, we follow straightforwardly the various well-known definitions of phase operator. First, a naive definition from (21) according to the correspondence principle would be

$$\hat{a}(t) = e^{i\hat{\theta}_{\mathrm{D}}(t)}\hat{n}^{1/2}(t), \quad \hat{a}^\dagger(t) = \hat{n}^{1/2}(t)e^{-i\hat{\theta}_{\mathrm{D}}(t)}, \tag{49}$$

which is the procedure taken by Dirac [4]. The $\hat{\theta}_{\mathrm{D}}$ formally satisfies the Lerner criterion (48), but only if one ignores the fact that to get Eq. (48) formally, one must multiply Eq. (35) by the singular operator $\hat{n}^{-1/2}$. Furthermore, $\hat{\theta}_{\mathrm{D}}(t)$ is not a Hermitian operator, so $e^{i\hat{\theta}_{\mathrm{D}}(t)}$ is not unitary [6]. Second, the phase operator by Susskind and Glogower [9] is defined as

$$\hat{E}(t) \equiv [\hat{n}(t) + 1]^{-1/2}\hat{a}(t), \quad \hat{E}^\dagger(t) \equiv \hat{a}^\dagger(t)[\hat{n}(t) + 1]^{-1/2}. \tag{50}$$

$\hat{E}$ and $\hat{E}^\dagger$ are the analogs of $e^{i\hat{\theta}}$ and $e^{-i\hat{\theta}}$, which satisfy the Lerner criterion (48). The analogs of $\cos\hat{\theta}$ and $\sin\hat{\theta}$ can be defined similarly. $\hat{E}$ and $\hat{E}^\dagger$ have the equivalent number state representation

$$\hat{E}(t) = \sum_{n=0}^{\infty} |n, t\rangle\langle n+1, t|, \quad \hat{E}^\dagger(t) = \sum_{n=0}^{\infty} |n+1, t\rangle\langle n, t|. \tag{51}$$

However, $\hat{E}$ is a one-sided unitary operator,

$$\hat{E}(t)\hat{E}^\dagger(t) = 1, \quad \hat{E}^\dagger(t)\hat{E}(t) = 1 - |0, t\rangle\langle 0, t|. \tag{52}$$

This is somewhat related with the ambiguity in defining a phase for $a(t) = 0$ in classical theory or for the vacuum state with $\hat{a}(t)|0, t\rangle = 0$ in quantum theory. Third, the phase operator defined by Pegg and Barnett in a finite dimensional subspace of Fock space [10], is now given by

$$\hat{\theta}_{\mathrm{PB}}(t) = \sum_{m=0}^{s} \theta_m |\theta_m, t\rangle\langle\theta_m, t|, \tag{53}$$

where

$$|\theta_m, t\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta_m} |n, t\rangle,$$
$$\theta_m = \theta_0 + \frac{2m\pi}{s+1}, \quad (\theta_0 = \text{constant}). \tag{54}$$

Fourth, the phase operator corresponding to the classical phase variable (22) is naively given by

$$\hat{\theta}_a(t) = \frac{i}{2}[\ln \hat{a}^\dagger(t) - \ln \hat{a}(t)]. \tag{55}$$

The phase operator (55) formally satisfies the commutation relation (47) and is a generalization of the time-independent one in Ref. [7] to the time-dependent one. However, it is not really well-defined, since $\hat{a}$ has a zero eigenstate, so that its logarithm is a divergent operator.

A few comments are in order. At present there is no consistent definition of phase operator for a quantum oscillator free from all conceptual and technical problems [6]. This dilemma is somewhat rooted in the quantization of phase variable according to Eq. (47):

$$\hat{\mathcal{I}} = i\hbar \frac{\partial}{\partial \theta}. \tag{56}$$

The eigenfunctions of Eq. (56), consistent with the periodicity of θ, are given by

$$\Psi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{-in\theta}, \quad (n = \text{all integers}). \tag{57}$$

The wave functions (57) form a basis of the Hilbert space on a circle and make the closure complete when n runs for all integers. Denoting the wave functions (57) by $|n, \theta\rangle$, the following phase operator [11]

$$e^{i\hat{\theta}} = \sum_{n=-\infty}^{\infty} |n, \theta\rangle\langle n+1, \theta|, \quad e^{-i\hat{\theta}} = \sum_{n=-\infty}^{\infty} |n+1, \theta\rangle\langle n, \theta| \tag{58}$$

becomes unitary

$$e^{i\hat{\theta}} e^{-i\hat{\theta}} = e^{-i\hat{\theta}} e^{i\hat{\theta}} = 1. \tag{59}$$

However, compatibility with Eqs. (45) restricts n to all non-negative integers. As Eq. (46) and (57) are the coordinate and phase-representations of the action eigenstate (45), the phase operator (58) is the same obtained by extending Eq. (51) over all integers including negative ones. But there is no physical meaning to the wave functions of oscillator with negative eigenvalues.

IV. CONCLUSION

In summary, we found pairs of real and complex invariants for a time-dependent classical oscillator, which are linear in position and momentum and transform canonically the phase space to invariant spaces. An action was defined that is quadratic in each invariant space, and its conjugate phase (angle) variable was found. The relation among different phase variables was exploited. Analogously, pairs of linear complex invariant operators were found for the time-dependent quantum oscillator. The action operator, which is the quantum analog of the classical action, may be used to construct a Fock space of exact quantum number eigenstates. As the invariant operators act as the annihilation and creation operators, we followed the procedure for time-independent oscillator to define the phase operator conjugate to the action operator for the time-dependent oscillator.

The action-phase variables were defined for the time-dependent classical oscillator exactly as for the time-independent oscillator. However any known scheme to define a consistent phase operator, even for the time-independent oscillator, confronts some technical difficulty [6]. The same degree of difficulty remains in defining the phase operator for the time-dependent oscillator. In this paper we did not intend to resolve all the puzzling issues of phase operator but to extend the current methods to the time-dependent oscillator. One interesting possibility will be to make use of the Wigner function of the exact quantum state. The Wigner function can be expressed in the invariant space, whose integral along each ray of constant phase may lead to a phase pseudo-probability distribution [12]. Then the phase probability distribution can be used to define an ensemble average of any physical function of the phase variable in quantum theory.

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